

BEURLING'S THEOREM FOR THE CLIFFORD-FOURIER TRANSFORM

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ABSTRACT. We give a generalization of Beurling's theorem for the Clifford-Fourier transform. Then, analogues of Hardy, Cowling-Price and Gelfand-Shilov theorems are obtained in Clifford analysis.

Keywords: Clifford analysis; Clifford-Fourier transform; Uncertainty principles; Beurling's theorem.

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1. INTRODUCTION

Uncertainty principle asserts that a function and its Fourier transform cannot both be sharply localized. In Eucliden spaces, many theorems are devoted to clarify it such as Beurling, Cowling and price, Hardy, Heisenberg.. Beurling theorem which is given by A. beurling [1] and proved by Hörmander [2] is the the most relevant one: that it gives Hardy, Cowling-Price and Gelfand-Shilov theroems.

Theorem 1.1. *Let $f \in L^2(\mathbb{R})$ be such that*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)| |\widehat{f}(y)| e^{|x||y|} dx dy < \infty,$$

then $f = 0$.

This theorem is generalized by Bonami et al [3] by giving solutions in terms of Hermite functions.

Theorem 1.2. *Let $N \geq 0$. Assume $f \in L^2(\mathbb{R}^m)$ satisfying*

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{|f(x)| |\widehat{f}(y)|}{(1 + |x| + |y|)^N} e^{|x||y|} dx dy < \infty,$$

then $f(x) = P(x)e^{-a|x|^2}$ where P is a polynomial of degree $< \frac{N-m}{2}$ and $a > 0$.

In 2010, Kawazoe and Majjaoli provide Beurling theorem for the Dunkl transform [4]. Moreover, Parui and Pusti give an alternative proof similar of that one used in [3] for the Dunkl transform (see [5]).

Our aim is to establish Beurling theorem for the Clifford-Fourier transform given by Brackx et al [6] and studied in [7, 8].

This paper is organized as follows. In section 2, we recall Clifford algebra and some notations that will be usefull in the sequel. In section 3, we remind the Clifford-Fourier transform and its properties. In section 4, we prove Beurling theorem for the Clifford-Fourier transform. Section 5 contains other uncertainty principles in Clifford analysis : Hardy, Cowling and Price and Gelfand Shilov.

2. NOTATIONS AND PRELIMINARIES

Clifford algebra $Cl_{0,m}$ over \mathbb{R}^m is defined as an algebra generated by the 2^m -dimensional basis:

$$(2.1) \quad \{1, e_1, e_2, e_3, \dots, e_m, e_{12}, \dots, e_{12..m}\},$$

where the multiplication of vectors from this basis is governed by the rules:

$$(2.2) \quad \begin{cases} e_i e_j = -e_j e_i, & \text{if } i \neq j; \\ e_i^2 = -1, & \forall 1 \leq i \leq m. \end{cases}$$

Clifford algebra $Cl_{0,m}$ is decomposed as :

$$(2.3) \quad Cl_{0,m} = \bigoplus_{k=0}^m Cl_{0,m}^k,$$

where $Cl_{0,m}^k = \text{span}\{e_{i_1} \dots e_{i_k}, i_1 < \dots < i_k\}$

A multivector x on the Clifford algebra can be presented by:

$$(2.4) \quad x = \sum_{A \in J} x_A e_A,$$

where $J := \{0, 1, \dots, m, 12, \dots, 12..m\}$, x_A real number and e_A belongs to the basis of $Cl_{0,m}$ defined above.

For each multivector x , the Clifford norm is:

$$(2.5) \quad \|x\|_c = \left(\sum_{A \in J} x_A^2 \right)^{\frac{1}{2}}.$$

Thus, a vector x in $Cl_{0,m}$ can be identify with

$$(2.6) \quad x = \sum_{i=1}^m x_i e_i,$$

and it's norm is

$$(2.7) \quad \|x\|_c^2 = \sum_{i=1}^m x_i^2.$$

We introduce the Dirac operator, Gamma operator and Laplace operator associated to a vector x respectively by:

$$(2.8) \quad \partial_x = \sum_{i=1}^m e_i \partial_{x_i};$$

$$(2.9) \quad \Gamma_x = - \sum_{j < k} e_j e_k (x_j \partial_{x_k} - x_k \partial_{x_j});$$

$$(2.10) \quad \Delta_c = \sum_{i=1}^m \partial_i^2.$$

In the Clifford algebra, a vector x and the Dirac operator satisfies :

$$(2.11) \quad ||x||_c^2 = -x^2$$

and

$$(2.12) \quad \Delta_c = -\partial_x^2.$$

The inner product and the wedge product of two vectors x and y are given respectively by:

$$(2.13) \quad \langle x, y \rangle := \sum_{j=1}^m x_j y_j = \frac{-1}{2}(xy + yx);$$

$$(2.14) \quad x \wedge y := \sum_{j < k} e_j e_k (x_j y_k - x_k y_j) = \frac{1}{2}(xy - yx).$$

Every function $f : \mathbb{R}^m \rightarrow Cl_{0,m}$ can be written as :

$$(2.15) \quad f(x) = f_0(x) + \sum_{i=1}^m e_i f_i(x) + \sum_{i < j} e_i e_j f_{ij}(x) + \dots + e_1 \dots e_m f_{1\dots m}(x),$$

where $f_0, f_i, \dots, f_{1\dots m}$ all real-valued functions.

We denote by :

- \mathcal{P}_k the space of homogeneous polynomials of degree k taking values in $Cl_{0,m}$,
- \mathcal{P} the space of polynomials taking values in $Cl_{0,m}$, i.e

$$P := \mathbb{R}[x_1, \dots, x_m] \otimes Cl_{0,m}.$$

- $\mathcal{M}_k := \text{Ker } \partial_x \cap \mathcal{P}_k$ the space of spherical monogenics of degree k ,
- $B(\mathbb{R}^m) \otimes Cl_{0,m}$ a class of integrable functions taking values in $Cl_{0,m}$ and satisfying

$$(2.16) \quad ||f||_B := \int_{\mathbb{R}^m} (1 + ||y||_c)^{\frac{m-2}{2}} ||f(y)||_c dy < \infty,$$

- $L^p(\mathbb{R}^m) \otimes Cl_{0,m}$ the space of integrable functions taking values in $Cl_{0,m}$ such that

$$(2.17) \quad ||f||_{p,c} = \left(\int_{\mathbb{R}^m} ||f(x)||_c^p dx \right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}^m} \left(\sum_{A \in J} (f_A(x))^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} < \infty,$$

where $J = \{0, 1, \dots, m, 12, 13, 23, \dots, 12\dots m\}$,

- $\mathcal{S}(\mathbb{R}^m)$ the Schwartz space of infinitely differentiable functions on \mathbb{R}^m which are rapidly decreasing as their derivatives.

3. CLIFFORD-FOURIER TRANSFORM

Definition 3.1. [8] *The Clifford-Fourier transform is defined on $B(\mathbb{R}^m) \otimes Cl_{0,m}$ by*

$$(3.1) \quad \mathcal{F}_\pm(f)(y) = (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} K_\pm(x, y) f(x) dx,$$

where

$$(3.2) \quad K_\pm(x, y) = e^{\mp i \frac{\pi}{2} \Gamma_y} e^{-i \langle x, y \rangle}.$$

Lemma 3.1. [9] *Let m be even. Then*

$$(3.3) \quad \|K_\pm(x, y)\|_c \leq C e^{\|x\|_c \|y\|_c}, \quad \forall x, y \in \mathbb{R}^m,$$

Theorem 3.2. *Let m be even and $f \in B(\mathbb{R}^m) \otimes Cl_{0,m}$. Then, there exists a positive constant A such that*

$$(3.4) \quad \|\mathcal{F}_\pm(f)(y)\|_c \leq C e^{\frac{\|y\|_c^2}{4}} \|f\|_B, \quad \forall \|y\|_c > A$$

$$(3.5) \quad \|\mathcal{F}_\pm(f)(y)\|_c \leq C(1 + A)^{\frac{m-2}{2}} \|f\|_B, \quad \forall \|y\|_c \leq A$$

Proof. Let $k(y) = (1 + \|y\|_c)^{\frac{m-2}{2}} e^{-\frac{\|y\|_c^2}{4}}$, $\forall y \in \mathbb{R}^m$.

Since $\lim_{\|y\|_c \rightarrow \infty} k(y) = 0$, there exists $A > 0$ such that for all $\|y\|_c > A$

$$\|k(y)\|_c \leq 1.$$

Thus, it follows that

$$(1 + \|y\|_c)^{\frac{m-2}{2}} \leq e^{\frac{\|y\|_c^2}{4}}, \quad \forall \|y\|_c > A.$$

Recall that the Clifford kernel [8] is written as :

$$K_-(x, y) = K_0^-(x, y) + \sum_{i < j} e_{ij} K_{ij}^-(x, y),$$

where $K_0^-(x, y)$ and $K_{ij}^-(x, y)$ satisfy

$$\begin{aligned} |K_0^-(x, y)| &\leq C(1 + \|x\|_c)^{\frac{m-2}{2}} (1 + \|y\|_c)^{\frac{m-2}{2}} \\ |K_{ij}^-(x, y)| &\leq C(1 + \|x\|_c)^{\frac{m-2}{2}} (1 + \|y\|_c)^{\frac{m-2}{2}}. \end{aligned}$$

We conclude. ■

Theorem 3.3. [7]

1) *The Clifford-Fourier transform is a continuous operator from $\mathcal{S}(\mathbb{R}^m) \otimes Cl_{0,m}$ to $\mathcal{S}(\mathbb{R}^m) \otimes Cl_{0,m}$.*

In particular, when m even, we have

$$\mathcal{F}_+ \mathcal{F}_+ = id_{\mathcal{S}(\mathbb{R}^m) \otimes Cl_{0,m}}.$$

2) *The Clifford-Fourier transform extends from $\mathcal{S}(\mathbb{R}^m) \otimes Cl_{0,m}$ to a continuous map on $L^2(\mathbb{R}^m) \otimes Cl_{0,m}$.*

In particular, when m even, we have

$$\|\mathcal{F}_\pm(f)\|_{2,c} = \|f\|_{2,c},$$

for all $f \in L^2(\mathbb{R}^m) \otimes Cl_{0,m}$.

Theorem 3.4. [9] Let $a > 0$ and $P \in \mathcal{P}_k(\mathbb{R}^m)$. Then, there exists $Q \in \mathcal{P}_k(\mathbb{R}^m)$ satisfying :

$$(3.6) \quad \mathcal{F}_\pm(P(\cdot)e^{-a\|\cdot\|_c^2})(x) = Q(x)e^{-\frac{\|x\|_c^2}{4a}}.$$

Definition 3.2. [8] Let m be even. The Clifford translation and the Clifford convolution for $f, g \in \mathcal{S}(\mathbb{R}^m)$ are introduced respectively by

$$(3.7) \quad T_y f(x) = (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} \overline{K_-(\epsilon, x)} K_-(y, \epsilon) \mathcal{F}(f)(\epsilon) d\epsilon,$$

$$(3.8) \quad f *_{Cl} g(x) = (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} T_y f(x) g(y) dy.$$

Theorem 3.5. [8] Let $f \in \mathcal{S}(\mathbb{R}^m) \otimes Cl_{0,m}$

i) For $m = 2$,

$$T_y f(x) = f(x - y)$$

ii) For m even and $m > 2$, we have

$$T_y f(x) = f_0(|x - y|),$$

for radial function f on \mathbb{R}^m , $f(x) = f_0(|x|)$ with $f_0 : \mathbb{R}_+ \rightarrow \mathbb{R}$.

Theorem 3.6. [8] Let $f \in \mathcal{S}(\mathbb{R}^m)$ be radial function and $g \in \mathcal{S}(\mathbb{R}^m) \otimes Cl_{0,m}$. Then,

$$\mathcal{F}_\pm(f *_{Cl} g) = \mathcal{F}_\pm(f) \mathcal{F}_\pm(g).$$

In particular, we have

$$f *_{Cl} g = g *_{Cl} f$$

4. BEURLING'S THEOREM FOR THE CLIFFORD-FOURIER TRANSFORM

In this section, we provide Beurling's theorem for the Clifford-Fourier transform.

Lemma 4.1. Assume $f \in L^2(\mathbb{R}^m) \otimes Cl_{0,m}$ satisfying

$$(4.1) \quad \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{\|f(x)\|_c \|\mathcal{F}_\pm(f)(y)\|_c}{(1 + \|x\|_c + \|y\|_c)^N} e^{\|x\|_c \|y\|_c} dx dy < \infty,$$

for some $N \geq 0$.

Then, $f \in B(\mathbb{R}^m) \otimes Cl_{0,m}$ and $\mathcal{F}_\pm(f) \in B(\mathbb{R}^m) \otimes Cl_{0,m}$.

Proof. We may suppose $f \neq 0$.

Applying Fubini's theorem, we obtain for almost every $y \in \mathbb{R}^m$,

$$\|\mathcal{F}_\pm(f)(y)\|_c \int_{\mathbb{R}^m} \frac{\|f(x)\|_c}{(1 + \|x\|_c + \|y\|_c)^N} e^{\|x\|_c \|y\|_c} dx < \infty.$$

Since $f \neq 0$, then $\mathcal{F}_\pm(f) \neq 0$. Thus, there exists $y_0 \neq 0$ such that $\mathcal{F}_\pm(f)(y_0) \neq 0$ and

$$(4.2) \quad \int_{\mathbb{R}^m} \frac{\|f(x)\|_c}{(1 + \|x\|_c)^N} e^{\|x\|_c \|y_0\|_c} dx < \infty.$$

Using (4.2) and the fact that for large x

$$\frac{e^{\|x\|_c \|y_0\|_c}}{(1 + \|x\|_c)^{N + \frac{m-2}{2}}} \geq 1,$$

it follows that

$$\int_{\mathbb{R}^m} (1 + \|x\|_c)^{\frac{m-2}{2}} \|f(x)\|_c dx < \infty.$$

Similarly, we get $\mathcal{F}_\pm(f) \in B(\mathbb{R}^m) \otimes Cl_{0,m}$. ■

Theorem 4.2. [10] *Let ϕ be an entire function of order 2 in the complex plane and let $\alpha \in]0, \frac{\pi}{2}[$. Assume that $|\phi(z)|$ is bounded by $C(1 + |z|)^N$ on the boundary of some angular sector $\{re^{i\beta} : r \geq 0, \beta_0 \leq \beta \leq \beta_0 + \alpha\}$. Then the same bound is valid inside the angular sector (when replacing C by $2^N C$).*

Theorem 4.3. *Let m be even and N be a positive integer. Assume $f \in L^2(\mathbb{R}^m) \otimes Cl_{0,m}$ such that*

$$(4.3) \quad \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{\|f(x)\|_c \|\mathcal{F}_\pm(f)(y)\|_c}{(1 + \|x\|_c + \|y\|_c)^N} e^{\|x\|_c \|y\|_c} dx dy < \infty.$$

Then,

$$f(x) = Q(x) e^{-a\|x\|_c^2}.$$

for some $a > 0$ and polynomial Q with degree less than $\frac{N-m}{2}$.

Proof. Step 1.

Let $g(x) = f *_{Cl} e^{-\frac{\|\cdot\|_c^2}{2}}(x)$.

By lemma 4.1, we have $f \in B(\mathbb{R}^m) \otimes Cl_{0,m}$. Thus, $g \in B(\mathbb{R}^m) \otimes Cl_{0,m}$.

Theorem 3.6 and theorem 3.4 yield

$$(4.4) \quad \mathcal{F}_\pm(g)(x) = \mathcal{F}_\pm(f)(x) \mathcal{F}_\pm(e^{-\frac{\|\cdot\|_c^2}{2}})(x) = \mathcal{F}_\pm(f)(x) e^{-\frac{\|x\|_c^2}{2}}.$$

We will show that g satisfies the following assumptions :

i)

$$\int_{\mathbb{R}^m} \|\mathcal{F}_\pm(g)(y)\|_c e^{\frac{\|y\|_c^2}{2}} dy < \infty,$$

ii)

$$\|\mathcal{F}_\pm(g)(y)\|_c \leq C e^{-\frac{\|y\|_c^2}{4}},$$

iii)

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{\|g(x)\|_c \|\mathcal{F}_\pm(g)(x)\|_c e^{\|x\|_c \|y\|_c}}{(1 + \|x\|_c + \|y\|_c)^N} dx dy < +\infty$$

4i)

$$\int_{\|x\|_c \leq R} \int_{\mathbb{R}^m} \|g(x)\|_c \|\mathcal{F}_\pm(g)(y)\|_c e^{\|x\|_c \|y\|_c} dx dy \leq C(1 + R)^N.$$

Since $\mathcal{F}(f) \in B(\mathbb{R}^m) \otimes Cl_{0,m}$, i) is a simple deduction from (4.4).

Let's prove ii).

Using (4.4) and theorem 3.2, it follows that

$$\|\mathcal{F}_\pm(g)(y)\|_c \leq C(1+A)^{\frac{m-2}{2}} \|f\|_B e^{-\frac{\|y\|_c^2}{2}}, \quad \forall \|y\|_c \leq A$$

and

$$\|\mathcal{F}_\pm(g)(y)\|_c \leq C\|f\|_B e^{-\frac{\|y\|_c^2}{4}}, \quad \forall \|y\|_c > A.$$

Thus, we get ii) where C is constant depending on f .

In order to establish iii), we use (4.4), theorem 3.5 and theorem 3.7.

Therefore, we find

$$\begin{aligned} I &:= \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{\|g(x)\|_c \|\mathcal{F}_\pm(g)(y)\|_c e^{\|x\|_c \|y\|_c}}{(1 + \|x\|_c + \|y\|_c)^N} dx dy \\ &\leq \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{\|f(t)\|_c e^{-\frac{\|x-t\|_c^2}{2}} \|\mathcal{F}_\pm(f)(y)\|_c e^{-\frac{\|y\|_c^2}{2}} e^{\|x\|_c \|y\|_c}}{(1 + \|x\|_c + \|y\|_c)^N} dt dx dy \\ &\leq \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \|f(t)\|_c \|\mathcal{F}_\pm(f)(y)\|_c A(t, y) e^{\|t\|_c \|y\|_c} dt dy, \end{aligned}$$

$$\text{with } A(t, y) := e^{-\frac{\|t\|_c^2}{2}} e^{-\frac{\|y\|_c^2}{2}} e^{-\|t\|_c \|y\|_c} \int_{\mathbb{R}^m} \frac{e^{-\frac{\|x\|_c^2}{2}} e^{\langle x, t \rangle} e^{\|x\|_c \|y\|_c}}{(1 + \|x\|_c + \|y\|_c)^N} dx.$$

We should prove that

$$(4.5) \quad A(t, y) \leq C(1 + \|t\|_c + \|y\|_c)^{-N}.$$

According to Cauchy-Schwarz's inequality, we have

$$|\langle x, t \rangle| \leq \|x\|_c \|t\|_c.$$

Thus

$$\int_{\mathbb{R}^m} \frac{e^{-\frac{\|x\|_c^2}{2}} e^{\langle x, t \rangle} e^{\|x\|_c \|y\|_c}}{(1 + \|x\|_c + \|y\|_c)^N} dx \leq e^{\frac{(\|t\|_c + \|y\|_c)^2}{2}} \int_{\mathbb{R}^m} \frac{e^{-\frac{(\|x\|_c - \|t\|_c - \|y\|_c)^2}{2}}}{(1 + \|x\|_c + \|y\|_c)^N} dx.$$

Hence

$$A(t, y) \leq \int_{\mathbb{R}^m} \frac{e^{-\frac{(\|x\|_c - \|t\|_c - \|y\|_c)^2}{2}}}{(1 + \|x\|_c + \|y\|_c)^N} dx.$$

Fix $0 < c < 1$. Let $B = (1 + \|t\|_c + \|y\|_c)$.

$$A(t, y) \leq \int_{\|x\|_c - \|t\|_c - \|y\|_c > cB} e^{-\frac{(\|x\|_c - \|t\|_c - \|y\|_c)^2}{2}} dx + \int_{\|x\|_c - \|t\|_c - \|y\|_c \leq cB} \frac{e^{-\frac{(\|x\|_c - \|t\|_c - \|y\|_c)^2}{2}}}{(1 + \|x\|_c + \|y\|_c)^N} dx.$$

If $\|x\|_c - \|t\|_c - \|y\|_c \leq cB$, then

$$\begin{aligned} 1 + \|x\|_c + \|y\|_c &\geq 1 + \frac{\|t\|_c}{2} - \frac{\|x\|_c - \|t\|_c}{2} + \|y\|_c \\ &\geq \frac{1}{2} + \frac{\|t\|_c}{2} + \frac{\|y\|_c}{2} - \frac{\|x\|_c - \|t\|_c - \|y\|_c}{2} \end{aligned}$$

$$\geq \frac{(1-c)}{2}B.$$

The proof of (4.5) and iii) is carried out by (4.3).

4i) Fix $k > 4$.

$$\begin{aligned} J &:= \int_{\|x\|_c \leq R} \int_{\mathbb{R}^m} \|g(x)\|_c \|\mathcal{F}_\pm(g)(y)\|_c e^{\|x\|_c \|y\|_c} dx dy \\ &= \int_{\|x\|_c \leq R} \|g(x)\|_c \left(\int_{\|y\| > kR} \|\mathcal{F}_\pm(g)(y)\|_c e^{\|x\|_c \|y\|_c} dy + \int_{\|y\| < kR} \|\mathcal{F}_\pm(g)(y)\|_c e^{\|x\|_c \|y\|_c} dy \right) dx. \end{aligned}$$

ii) implies

$$\begin{aligned} J &\leq \int_{\|x\|_c \leq R} \|g(x)\|_c \left(\int_{\|y\| > kR} C e^{-(\frac{1}{4} - \frac{1}{k})\|y\|_c^2} dy + \int_{\|y\| < kR} \|\mathcal{F}_\pm(g)(y)\|_c e^{\|x\|_c \|y\|_c} dy \right) dx \\ &\leq C \|g\|_{1,c} + \int_{\|x\|_c \leq R} \int_{\|y\| < kR} \|g(x)\|_c \|\mathcal{F}_\pm(g)(y)\|_c e^{\|x\|_c \|y\|_c} dy dx. \end{aligned}$$

Multiplying and dividing by $(1 + \|x\|_c + \|y\|_c)^N$ in the integral of right side, we obtain

$$J \leq C(1+R)^N \int_{\|x\|_c \leq R} \int_{\|y\| < kR} \frac{\|g(x)\|_c \|\mathcal{F}_\pm(g)(y)\|_c e^{\|x\|_c \|y\|_c}}{(1 + \|x\|_c + \|y\|_c)^N} dx dy.$$

iii) completes the proof of 4i).

Step 2. Combining theorem 3.3, lemma 3.1 and ii), we get g admits an holomorphic extension to $\mathbb{C} \otimes \mathbb{R}^m$. Moreover, for all $z \in \mathbb{C} \otimes \mathbb{R}^m$

$$\begin{aligned} \|g(z)\|_c &= \|\mathcal{F}_\pm \circ \mathcal{F}_\pm(g)(z)\|_c \\ &\leq (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} e^{\|y\|_c \|z\|_c} \|\mathcal{F}_\pm(g)(y)\|_c dy \\ &\leq (2\pi)^{-\frac{m}{2}} C \int_{\mathbb{R}^m} e^{\|y\|_c \|z\|_c} e^{-\frac{\|y\|_c^2}{4}} dy \\ &\leq C e^{\|z\|_c^2}. \end{aligned}$$

Thus, g is entire of order 2.

We should prove that for all $\mathbb{C} \otimes \mathbb{R}^m$, $g(z)g(iz)$ is a polynomial.

Using theorem 3.3 and lemma 3.1, it follows that for all $x \in \mathbb{R}^m$ and $\theta \in \mathbb{R}$

$$(4.6) \quad \|g(e^{i\theta}x)\|_c \leq C \int_{\mathbb{R}^m} e^{\|x\|_c \|y\|_c} \|\mathcal{F}_\pm(g)(y)\|_c dy.$$

Let

$$\Gamma(z) = \int_0^{z_1} \dots \int_0^{z_m} g(u)g(iu)du.$$

Following the proof of [3], we conclude. ■

5. APPLICATIONS TO OTHER UNCERTAINTY PRINCIPLES

In this section, we show the relevance of Beurling theorem since it entail Hardy, Cowling-Price and Gelfand Shilov theorems.

Corollary 5.1 (Hardy theorem). *Let m be even. Assume that $f \in L^2(\mathbb{R}^m) \otimes Cl_{0,m}$ satisfies*

$$(5.1) \quad \|f(x)\|_c \leq C(1 + \|x\|_c)^N e^{-a\|x\|_c^2}$$

and

$$(5.2) \quad \|\mathcal{F}(f)(y)\|_c \leq C(1 + \|y\|_c)^N e^{-b\|y\|_c^2},$$

for some $N \in \mathbb{N}$. Then, three cases can occur

i) If $ab > \frac{1}{4}$, then $f = 0$.

ii) If $ab = \frac{1}{4}$, then $f = P(x)e^{-a\|x\|_c^2}$ with $\deg P \leq N$.

iii) If $ab < \frac{1}{4}$, there are many functions satisfying these estimates.

Corollary 5.2 (Cowling and Price theorem). *Let $N \in \mathbb{N}$, $1 < p, q < \infty$ and m be even. Let $f \in L^2(\mathbb{R}^m) \otimes Cl_{0,m}$ be such that*

$$(5.3) \quad \int_{\mathbb{R}^m} \frac{e^{\alpha p\|x\|_c^2} \|f(x)\|_c^p}{(1 + \|x\|_c)^N} dx < \infty$$

$$(5.4) \quad \int_{\mathbb{R}^m} \frac{e^{\beta q\|y\|_c^2} \|\mathcal{F}(f)(y)\|_c^q}{(1 + \|y\|_c)^N} dy < \infty$$

with $\alpha\beta \geq \frac{1}{4}$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

i) $f = 0$, if $\alpha\beta > \frac{1}{4}$.

ii) $f = P(x)e^{-\alpha\|x\|_c^2}$ with P is a polynomial of degree $< \min\{\frac{N-m}{p}, \frac{N-m}{q}\}$, if $\alpha\beta = \frac{1}{4}$.

Corollary 5.3 (Gelfand Shilov theorem). *Let $N \in \mathbb{N}$, $1 < p, q < \infty$, $\alpha, \beta > 0$ and m be even. Let $f \in L^2(\mathbb{R}^m) \otimes Cl_{0,m}$ satisfy*

$$(5.5) \quad \int_{\mathbb{R}^m} \frac{\|f(x)\|_c e^{\frac{(2\alpha\|x\|_c)^p}{p}}}{(1 + \|x\|_c)^N} dx < \infty$$

$$(5.6) \quad \int_{\mathbb{R}^m} \frac{\|\mathcal{F}(f)(y)\|_c e^{\frac{(2\beta\|y\|_c)^q}{q}}}{(1 + \|y\|_c)^N} dy < \infty$$

with $\alpha\beta \geq \frac{1}{4}$ and $\frac{1}{p} + \frac{1}{q} = 1$. We have the following results :

i) If $\alpha\beta > \frac{1}{4}$ or $(p, q) \neq (2, 2)$, then $f = 0$.

ii) If $\alpha\beta = \frac{1}{4}$ and $p = q = 2$, then $f = P(x)e^{-2\alpha^2\|x\|_c^2}$ where P is a polynomial with degree less than $N - m$.

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